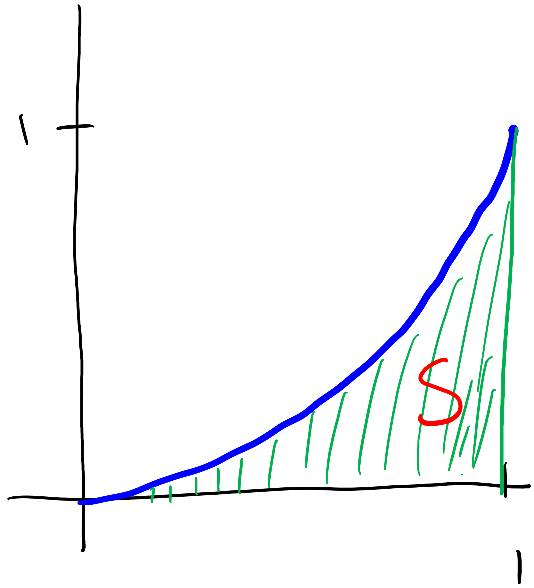
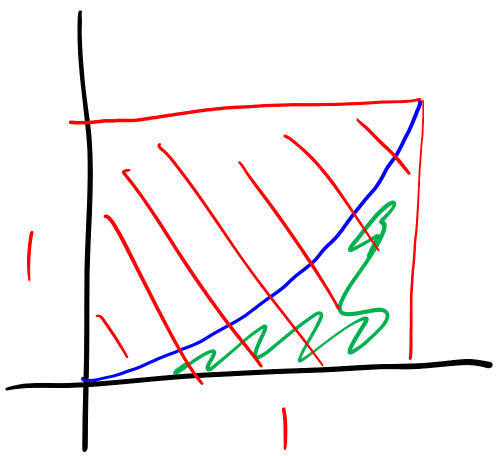
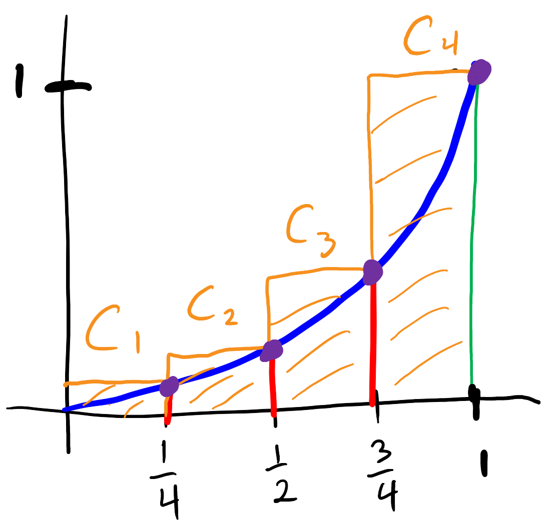
Consider $f(x) = x^2$ on $[0,1]$.



What is the area of the shaded region S?



Area ≈ 1



Area under $f(x) = x^2$ is approximately $C_1 + C_2 + C_3 + C_4$

$$R_4 = C_1 + C_2 + C_3 + C_4$$

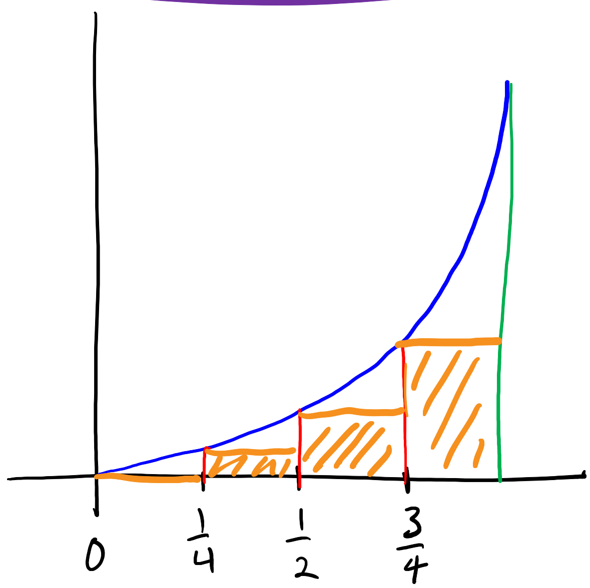
Right-handed approximation

$$= \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} \cdot f\left(\frac{1}{2}\right) + \frac{1}{4} \cdot f\left(\frac{3}{4}\right) + \frac{1}{4} \cdot f(1)$$

$$= \frac{1}{4}\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1)\right)$$

$$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16} + 1\right) = \frac{15}{32}$$

Left Handed Approximation

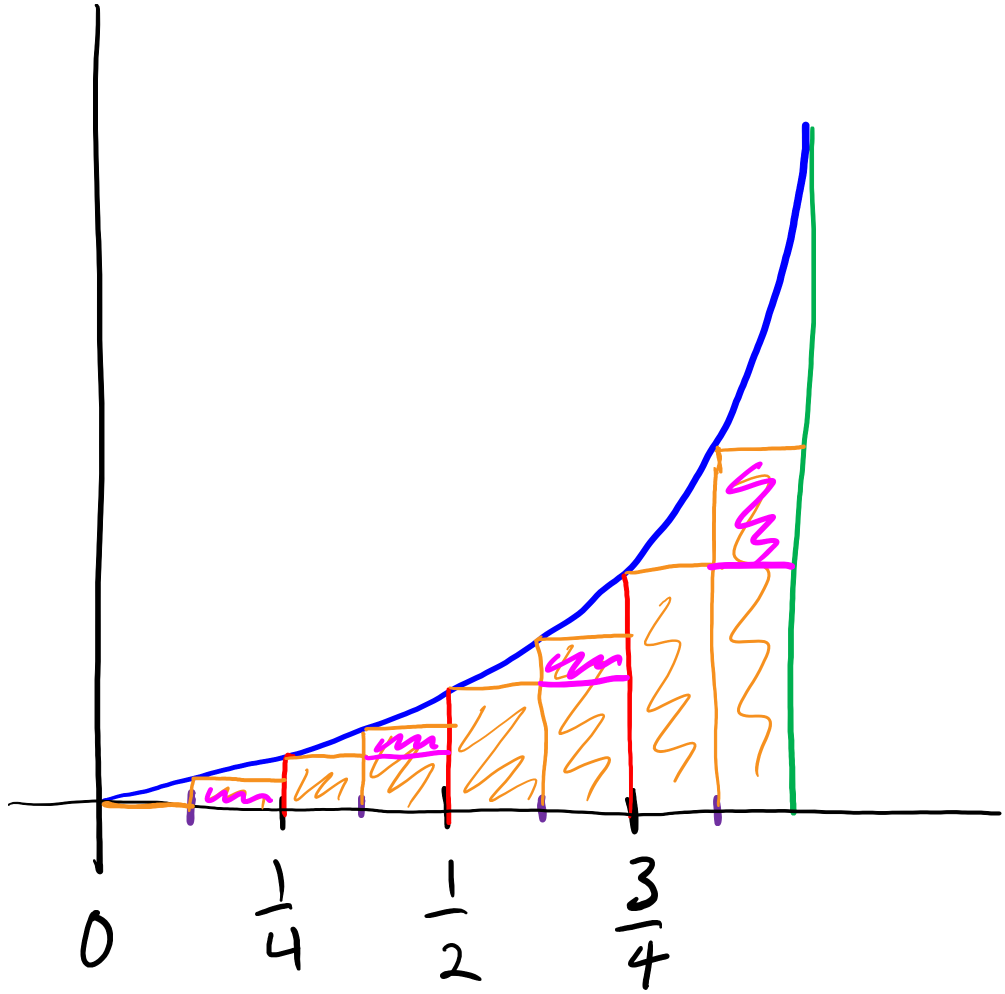


$$L_4 = \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right) = \frac{7}{32}$$

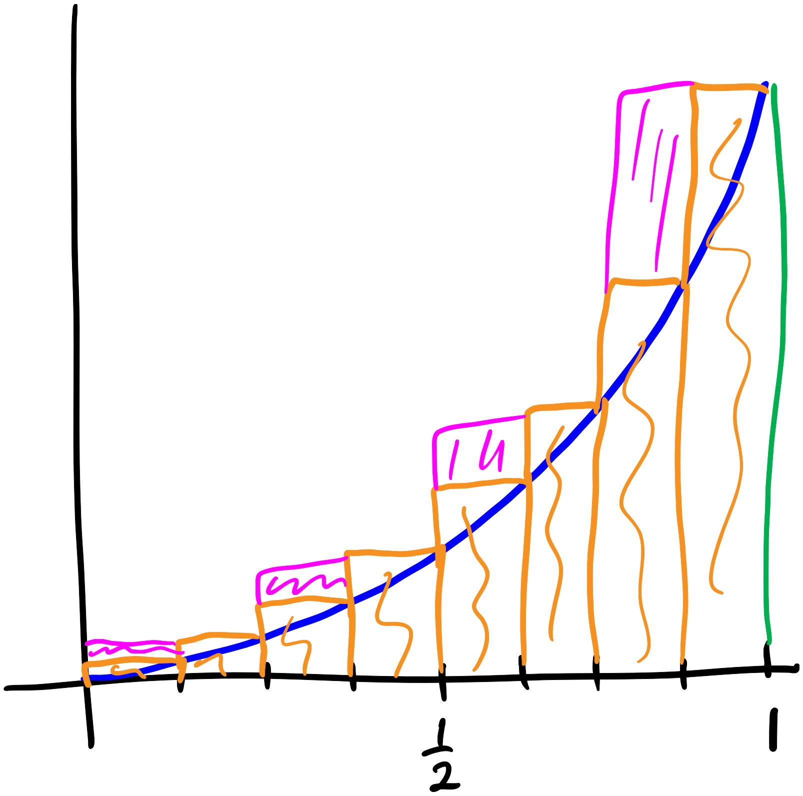
$$\frac{7}{32} < \text{True Area} < \frac{15}{32}$$

$$0.21875 < A < 0.46875$$



$$L_8 = \frac{1}{8}\left(f(0) + f\left(\tfrac{1}{8}\right) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{8}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{5}{8}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{7}{8}\right)\right)$$

$$= 0.2734375$$



$R_8 = 0.3984375$

$\sim .25$

4-rectangles: $0.21875 < A < 0.46875$

8-rectangles: $0.2734375 < A < 0.3984375$

$\sim .12$

Use n-rectangles:

width of each rectangle $= \frac{1}{n}$

$$R_n = \frac{1}{n}\left(f\left(\tfrac{1}{n}\right) + f\left(\tfrac{2}{n}\right) + f\left(\tfrac{3}{n}\right) + \cdots + f\left(\tfrac{n-1}{n}\right) + f\left(\tfrac{n}{n}\right)\right)$$

$$= \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2} + \frac{n^2}{n^2}\right)$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2\right)$$

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$2n^3 + \cdots \ \cdots$

$$\text{Actual area} = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}$$

$$1 + 2 + 3 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$